数理解析研究所講究録
第 1755 巻 2011 年 210-217

EXISTENCE THEOREMS FOR SADDLE POINTS OF SET-VALUED MAPS VIA NONLINEAR SCALARIZATION METHODS*
(非線形スカラー化手法を用いた集合値写像の鞍点の存在定理)

新潟大学・大学院自然科学研究科 桑野一成, 田中 環, 山田修司
Issei Kuwano, Tamaki Tanaka, Syuuji Yamada†
Graduate School of Science and Technology,
Niigata University, Japan

Abstract

In the paper, we introduce five types of concepts for saddle points of set-valued maps and show existence theorems for these saddle points by using nonlinear scalarizing functions for sets introduced by Kuwano, Tanaka, and Yamada in 2009.

1 Introduction

Let X and Y be two real topological vector spaces, F a map on $X \times Y$. In real-valued case, $(x_0, y_0) \in X \times Y$ is a saddle point of F if

$$F(x_0, y) \leq F(x_0, y_0) \leq F(x, y_0)$$

for any $x \in X$ and $y \in Y$. In vector-valued case, a saddle point $(x_0, y_0) \in X \times Y$ with respect to partial ordering $\leq_C$ induced by a convex cone C is defined by

$$F(x, y_0) \not\leq_C F(x_0, y_0) \not\leq_C F(x_0, y)$$

for any $x \in X$ and $y \in Y$, and it is called C-saddle point of F. Many researchers have been investigated existence theorems for saddle points and C-saddle points. In [7] and [8], we consider five types of generalizations for C-saddle points and investigate sufficient conditions for the existence of these saddle points by using nonlinear scalarization methods for sets proposed in [4].

The aim of the paper is to introduce three types of existence theorems for cone saddle points of set-valued maps.

The organization of the paper is as follows. In Section 2, we review mathematical methodology proposed in [3] on comparison between two sets in an ordered vector space and some basic concepts of set-valued optimization. In Section 3, we consider two types of nonlinear scalarizing functions for sets proposed by the unified approach in [4], and

* This work is based on research 21540121 supported by Grant-in-Aid for Scientific Research (C) from Japan Society for the Promotion of Science.

† *E-mail:* kuwano@m.sc.niigata-u.ac.jp, {tamaki,yamada}@math.sc.niigata-u.ac.jp

2000 *Mathematics Subject Classification.* 49J53, 54C60, 90C46, 90C47.
Key words and phrases. Set-valued analysis, set-valued optimization, nonlinear scalarization, minimax theorems.

investigate their properties. In Section 4, we introduce five types of concepts for cone saddle points of set-valued maps, and three types of existence theorems for these saddle points proved in [7, 8].

2 Mathematical Preliminaries

Throughout the paper, X and Y are two real topological vector spaces and C is a proper closed convex cone in Y (that is, $C \neq Y$, $C + C = C$ and $\lambda C \subset C$ for all $\lambda \geq 0$) with nonempty topological interior. We define a partial ordering $\leq_C$ on Y as follows:

$$x \leq_C y \quad \text{if} \quad y - x \in C \quad \text{for} \quad x, y \in Y.$$

Let F be a set-valued map from $S \subset X$ into 2^Y where $S := \{x \in X | F(x) \neq \emptyset\}$ and assume that S is a convex set. For $A \in 2^Y \setminus \{\emptyset\}$, we denote the topological interior of A by int$A$. Also, we denote the algebraic sum, algebraic difference of A and C by $A + C := \bigcup_{a \in A}(a + C)$, $A - C := \bigcup_{a \in A}(a - C)$, respectively. In addition, we denote the composite function of two functions f and g by $g \circ f$. When $x \leq_C y$ for $x, y \in Y$, we define the order interval between x and y by $[x, y] := \{z \in Y | x \leq_C z \,\text{and}\, z \leq_C y\}$.

At first, we review some basic concepts of set-relation.

Definition 2.1. (See Ref. [3].) For any $A, B \in 2^Y \setminus \{\emptyset\}$ and convex cone C in Y, we write

$A \leq_C^{(1)} B$ by $A \subset \bigcap_{b \in B}(b - C)$, equivalently $B \subset \bigcap_{a \in A}(a + C)$,
$A \leq_C^{(2)} B$ by $A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$,
$A \leq_C^{(3)} B$ by $B \subset (A + C)$,
$A \leq_C^{(4)} B$ by $\left(\bigcap_{a \in A}(a + C)\right) \cap B \neq \emptyset$,
$A \leq_C^{(5)} B$ by $A \subset (B - C)$,
$A \leq_C^{(6)} B$ by $A \cap (B - C) \neq \emptyset$, equivalently $(A + C) \cap B \neq \emptyset$.

Proposition 2.1. (See [3].) *For any $A, B \in 2^Y \setminus \{\emptyset\}$, the following statements hold:*

$$\begin{array}{ll} A \leq_C^{(1)} B \text{ implies } A \leq_C^{(2)} B, & A \leq_C^{(1)} B \text{ implies } A \leq_C^{(4)} B, \\ A \leq_C^{(2)} B \text{ implies } A \leq_C^{(3)} B, & A \leq_C^{(4)} B \text{ implies } A \leq_C^{(5)} B, \\ A \leq_C^{(3)} B \text{ implies } A \leq_C^{(6)} B, & A \leq_C^{(5)} B \text{ implies } A \leq_C^{(6)} B. \end{array}$$

Proposition 2.2. (See [4].) *For any $A, B \in 2^Y \setminus \{\emptyset\}$, the following statements hold:*

(i) *For each $j = 1, \ldots, 6$,*
$A \leq_C^{(j)} B$ implies $(A + y) \leq_C^{(j)} (B + y)$ for $y \in Y$, and
$A \leq_C^{(j)} B$ implies $\alpha A \leq_C^{(j)} \alpha B$ for $\alpha \geq 0$.

(ii) *For each $j = 1, \ldots, 5$, $\leq_C^{(j)}$ is transitive.*

(iii) *For each $j = 3, 5, 6$, $\leq_C^{(j)}$ is reflexive.*

From (b) and (c) of Proposition 2.2, $\leq_C^{(6)}$ is difficult to say as order. Hence, we consider mainly the cases of $j = 1, \ldots, 5$ in the paper.

By using the set-relations defined in Definition 2.1, we consider the following five kinds of set-valued optimization problems with $j = 1, \dots, 5$:

$$(j\text{-SVOP})\begin{cases} j\text{-Optimize} & F(x) \\ \text{Subject to} & x \in S. \end{cases}$$

Then, we introduce some concepts of solutions for (j-SVOP). Let $x_0 \in S$. For each $j = 1, \dots, 5$, x_0 is a *minimal solution of* (j-SVOP) if for any $x \in S \setminus \{x_0\}$,

$$F(x) \leq_C^{(j)} F(x_0) \quad \text{implies} \quad F(x_0) \leq_C^{(j)} F(x); \tag{2.1}$$

and x_0 is a *maximal solution of* (j-SVOP) if for any $x \in S \setminus \{x_0\}$,

$$F(x_0) \leq_C^{(j)} F(x) \quad \text{implies} \quad F(x) \leq_C^{(j)} F(x_0). \tag{2.2}$$

If C is replaced by int$C$, then x_0 is a *weak minimal solution* (resp., *weak maximal solution*) of (j-SVOP). We denote the family of sets satisfying (2.1) (resp., (2.2)) by $\text{Min}_{(j)}F(S)$ (resp., $\text{Max}_{(j)}F(S)$) and the case of weak minimal (resp., weak maximal) by $\text{WMin}_{(j)}F(S)$ (resp., $\text{WMax}_{(j)}F(S)$) where $F(S) = \{F(x) | x \in S\}$. It is clear that if x_0 is a minimal (resp., maximal) solution of (j-SVOP) then x_0 is a weak minimal (resp., weak maximal) solution of (j-SVOP).

Let us recall some definitions of C-notions (see [2].) A subset A of Y is said to be *C-convex* (resp., *C-closed*) if $A + C$ is convex (resp., closed). Moreover, we say that F is *C-notion on S* if $F(x)$ has the property C-notion for every $x \in S$.

Next, we introduce several definitions of C-convexity and C-continuity for set-valued maps. These notions are used in Sections 3 and 4.

Definition 2.2. (See [4].) For each $j = 1, \dots, 5$,

(i) F is called a *type (j) naturally quasi C-convex function* if for each $x, y \in S$ and $\lambda \in (0, 1)$, there exists $\mu \in [0, 1]$ such that

$$F(\lambda x + (1 - \lambda)y) \leq_C^{(j)} \mu F(x) + (1 - \mu)F(y).$$

(ii) F is called a *type (j) naturally quasi C-concave function* if for each $x, y \in S$ and $\lambda \in (0, 1)$, there exists $\mu \in [0, 1]$ such that

$$\mu F(x) + (1 - \mu)F(y) \leq_C^{(j)} F(\lambda x + (1 - \lambda)y).$$

Definition 2.3. (See [8].) For each $j = 1, \dots, 5$,

(i) F is called a *type (j) C-convexlike function* if for every $x, y \in S$ and $\lambda \in (0, 1)$, there exists $z \in S$ such that

$$F(z) \leq_C^{(j)} \lambda F(x) + (1 - \lambda)F(y).$$

(ii) F is called a *type (j) C-concavelike function* if for every $x, y \in S$ and $\lambda \in (0, 1)$, there exists $z \in S$ such that

$$\lambda F(x) + (1 - \lambda)F(y) \leq_C^{(j)} F(z).$$

Definition 2.4. (See [2].) Let $x \in S$. Then,

(i) F is called *C-lower continuous at x* if for every open set V with $F(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x such that $F(y) \cap (V + C) \neq \emptyset$ for all $y \in U$. We shall say that F is *C-lower continuous on S* if it is C-lower continuous at every point $x \in S$,

(ii) F is called *C-upper continuous at x* if for every open set V with $F(x) \subset V$, there exists an open neighborhood U of x such that $F(y) \subset V + C$ for all $y \in U$. We shall say that F is *C-upper continuous on S* if it is C-upper continuous at every point $x \in S$.

3 Unified Types of Scalarizing Functions for Sets

In [4], we propose the following nonlinear scalarizing functions for sets: Let $V, V' \in 2^Y \setminus \{\emptyset\}$, and direction $k \in \mathrm{int}C$. For each $j = 1, \ldots, 5$, we define $I^{(j)}_{k,V'} : 2^Y \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$I^{(j)}_{k,V'}(V) := \inf \left\{ t \in \mathbb{R} \;\middle|\; V \leq^{(j)}_C (tk + V') \right\}.$$

In this section, we introduce some properties of these functions and several sufficient conditions for the existence of solutions of (j-SVOP).

Proposition 3.1. (See [6].) *Let $A, B \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(i) *If $A \leq^{(1)}_C B$, A is $(-C)$-closed and B is C-closed then*

$$I^{(1)}_{k,V'}(A) < I^{(1)}_{k,V'}(B).$$

(ii) *For each $j = 2, 3$, if $A \leq^{(j)}_{\mathrm{int}C} B$ and B is C-closed then*

$$I^{(j)}_{k,V'}(A) < I^{(j)}_{k,V'}(B).$$

(iii) *For each $j = 4, 5$, if $A \leq^{(j)}_{\mathrm{int}C} B$ and A is $(-C)$-closed then*

$$I^{(j)}_{k,V'}(A) < I^{(j)}_{k,V'}(B).$$

Next, we introduce certain inherited properties on cone-convexity and cone-continuity of set-valued maps proved in [4, 5, 8, 10].

Lemma 3.1. (See [4, 5].) *Let $k \in \mathrm{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(i) *For each $j = 1, 2, 3$, if F is type (j) naturally quasi C-convex, then $I^{(j)}_{k,V'} \circ F$ is quasi convex. Moreover, if F is type (j) naturally quasi C-concave, then $I^{(j)}_{k,V'} \circ F$ is quasi concave.*

(ii) *For each $j = 4, 5$, if F is type (j) naturally quasi C-convex and V' is $(-C)$-convex, then $I^{(j)}_{k,V'} \circ F$ is quasi convex. Moreover, if F is type (j) naturally quasi C-concave and V' is $(-C)$-convex, then $I^{(j)}_{k,V'} \circ F$ is quasi concave.*

Lemma 3.2. (See [8].) *Let $k \in \mathrm{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(i) *For each $j = 1, 2, 3$, if F is type (j) C-convexlike and V' is C-convex, then $I^{(j)}_{k,V'} \circ F$ is convexlike.*
(ii) *For each $j = 4, 5$, if F is type (j) C-convexlike and V' is $(-C)$-convex, then $I^{(j)}_{k,V'} \circ F$ is convexlike.*

Lemma 3.3. (See [8].) *Let $k \in \mathrm{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(i) *For each $j = 1, 2, 3$, if F is type (j) C-concavelike and V' is C-convex, then $I^{(j)}_{k,V'} \circ F$ is concavelike.*
(ii) *For each $j = 4, 5$, if F is type (j) C-concavelike and V' is $(-C)$-convex, then $I^{(j)}_{k,V'} \circ F$ is concavelike.*

Lemma 3.4. (See [10].) *Let $k \in \mathrm{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(i) *For each $j = 1, 4, 5$, if F is C-lower continuous on S then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on S. Moreover, if F is $(-C)$-upper continuous on S then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on S.*
(ii) *For each $j = 2, 3$, if F is $(-C)$-lower continuous on S then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on S. Moreover, if F is C-upper continuous on S then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on S.*

Let $V' \in 2^Y \setminus \{\emptyset\}$ and direction $k \in \mathrm{int}C$. To show sufficient conditions for the existence of solutions of (j-SVOP) by using properties of $I^{(j)}_{k,V'}$, we consider the following two kinds of scalar optimization problems:

$$\inf_{x \in S} (I^{(j)}_{k,V'} \circ F)(x) \quad \text{and} \quad \sup_{x \in S} (I^{(j)}_{k,V'} \circ F)(x).$$

Lemma 3.5. (See [7].) *Assume that F is C-closed on S and $x_0 \in S$. Let $k \in \mathrm{int}C$. For each $j = 1, 2, 3$, the following statements hold:*

(i) *If x_0 is a solution of $\inf_{x \in S} (I^{(j)}_{k,V'} \circ F)(x)$, then x_0 is a weak minimal solution of (j-SVOP).*
(ii) *If x_0 is a solution of $\sup_{x \in S} (I^{(j)}_{k,V'} \circ F)(x)$, then x_0 is a weak maximal solution of (j-SVOP).*

Lemma 3.6. (See [7].) *Assume that F is $(-C)$-closed on S and $x_0 \in S$. Let $k \in \mathrm{int}C$. For each $j = 4, 5$, the following statements hold:*

(i) *If x_0 is a solution of $\inf_{x \in S} (I^{(j)}_{k,V'} \circ F)(x)$, then x_0 is a weak minimal solution of (j-SVOP).*
(ii) *If x_0 is a solution of $\sup_{x \in S} (I^{(j)}_{k,V'} \circ F)(x)$, then x_0 is a weak maximal solution of (j-SVOP).*

4 Existence Theorems for Saddle Points of Set-Valued Maps

At first, we introduce definitions of saddle points for set-valued maps proposed in [8]. For each $j = 1, \ldots, 5$, if $(x_0, y_0) \in X \times Y$ satisfies the following properties:

$$\text{(i)} \quad F(x, y_0) \leq_C^{(j)} F(x_0, y_0) \quad \text{implies} \quad F(x_0, y_0) \leq_C^{(j)} F(x, y_0),$$

$$\text{(ii)} \quad F(x_0, y_0) \leq_C^{(j)} F(x_0, y) \quad \text{implies} \quad F(x_0, y) \leq_C^{(j)} F(x_0, y_0),$$

for any $x \in X$ and $y \in Y$, then we call it type (j) *C-saddle point* of F. It is equivalent to

$$F(x_0,\, y_0) \in \left\{\mathrm{Min}_{(j)} F(X, y_0)\right\} \cap \left\{\mathrm{Max}_{(j)} F(x_0, Y)\right\}.$$

If C is replaced by int$C$ then we call it type (j) *weak C-saddle point* of F.

In this section, we give three types of existence theorems for type (j) cone saddle points of set-valued maps. At first, we introduce the first existence theorems which are natural extensions of Sion's minimax theorem (see [9]).

Theorem 4.1. (See [7].) *Let X and Y be nonempty compact convex subsets of two real topological vector spaces, respectively, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, V' a nonempty subset of Z and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that F is C-closed and $(-C)$-closed on $X \times Y$. If F satisfies the following conditions:*

(i) *$x \to F(x, y)$ is C-lower continuous and type (1) naturally quasi C-convex on X for every $y \in Y$,*
(ii) *$x \to F(x, y)$ is $(-C)$-upper continuous and type (1) naturally quasi C-concave on Y for every $x \in X$,*

then F has at least one type (1)-weak saddle point.

Theorem 4.2. (See [7].) *Let X and Y be nonempty compact convex subsets of two real topological vector spaces, respectively, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, V' a nonempty subset of Z and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that F is C-closed on $X \times Y$. For each $j = 2, 3$, if F satisfies that*

(i) *$x \to F(x, y)$ is C-upper continuous and type (j) naturally quasi C-convex on X for every $y \in Y$,*
(ii) *$x \to F(x, y)$ is $(-C)$-lower continuous and type (j) naturally quasi C-concave on Y for every $x \in X$,*

then F has at least one type (j)-weak saddle point.

Theorem 4.3. (See [7].) *Let X and Y be nonempty compact convex subsets of two real topological vector spaces, respectively, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, V' a nonempty subset of Z and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that F is $(-C)$-closed on $X \times Y$ and V' is $(-C)$-convex. For each $j = 4, 5$, if F satisfies that*

(i) *$x \to F(x,\, y)$ is C-lower continuous and type (j) naturally quasi C-convex on X for every $y \in Y$,*
(ii) *$x \to F(x,\, y)$ is $(-C)$-upper continuous and type (j) naturally quasi C-concave on Y for every $x \in X$,*

then F has at least one type (j)-weak saddle point.

Next, we introduce the second existence theorems which are natural extensions of Fan type minimax theorem (see [1]).

Theorem 4.4. (See [8].) *Let X be a nonempty compact subset of real topological space, Y any space, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, V' a nonempty subset of Z and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that F is C-closed and $(-C)$-closed on $X \times Y$. If F satisfies that*

(i) *$x \to F(x, y)$ is type (1) C-convexlike on X for every $y \in Y$,*
(ii) *$x \to F(x, y)$ is $(-C)$-upper continuous and type (1) C-concavelike on Y for every $x \in X$,*

then F has at least one type (1)-weak saddle point.

Theorem 4.5. (See [8].) *Let X be a nonempty compact subset of real topological space, Y any space, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, V' a nonempty subset of Z and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that F is C-closed on $X \times Y$. For each $j = 2, 3$, if F satisfies that*

(i) *$x \to F(x, y)$ is type (j) C-convexlike on X for every $y \in Y$,*
(ii) *$x \to F(x, y)$ is $(-C)$-lower continuous and type (j) C-concavelike on Y for every $x \in X$,*

then F has at least one type (j)-weak saddle point.

Theorem 4.6. (See [8].) *Let X be a nonempty compact subset of real topological space, Y any space, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, V' a nonempty subset of Z and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that F is $(-C)$-closed on $X \times Y$. For each $j = 4, 5$, if F satisfies that*

(i) *$x \to F(x, y)$ is type (j) C-convexlike on X for every $y \in Y$,*
(ii) *$x \to F(x, y)$ is $(-C)$-upper continuous and type (j) C-concavelike on Y for every $x \in X$,*

then F has at least one type (j)-weak saddle point.

Finally, we give the third existence theorems for type (j) cone saddle points of set-valued maps with separated form.

Theorem 4.7. (See [7].) *Let X and Y be nonempty compact subsets of two real topological spaces, respectively, Z a real ordered topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, V' a nonempty subset of Z and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. If F satisfies that*

(i) *$F(x, y) := G_1(x) \cup G_2(y)$,*
(ii) *G_1 is C-closed and C-lower continuous on X,*
(iii) *G_2 is $(-C)$-closed and $(-C)$-upper continuous on Y,*

where $G_1 : X \to 2^Z \setminus \{\emptyset\}$ and $G_2 : Y \to 2^Z \setminus \{\emptyset\}$, then F has at least one type (1) C-saddle point.

Theorem 4.8. (See [7].) *Let X and Y be nonempty compact subsets of two real topological spaces, respectively, Z a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, V' a nonempty subset of Z and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. For each $j = 2, 3$, if F satisfies that*

(i) $F(x,y) := G_1(x) \cup G_2(y)$,
(ii) G_1 *is* C*-closed and* C*-upper continuous on* X,
(iii) G_2 *is* C*-closed and* $(-C)$*-lower continuous on* Y,

where $G_1 : X \to 2^Z \setminus \{\emptyset\}$ *and* $G_2 : Y \to 2^Z \setminus \{\emptyset\}$, *then* F *has at least one type* (j) *weak* C*-saddle point.*

Theorem 4.9. (See [7].) *Let* X *and* Y *be nonempty compact subsets of two real topological spaces, respectively,* Z *a real topological vector space with the partial ordering* $\leq_C$, $k \in \mathrm{int}C$, V' *a nonempty subset of* Z *and* $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. *For each* $j = 4, 5$, *if* F *satisfies that*

(i) $F(x,y) := G_1(x) \cup G_2(y)$,
(ii) G_1 *is* $(-C)$*-closed and* C*-lower continuous on* X,
(iii) G_2 *is* $(-C)$*-closed and* $(-C)$*-upper continuous on* Y,

where $G_1 : X \to 2^Z \setminus \{\emptyset\}$ *and* $G_2 : Y \to 2^Z \setminus \{\emptyset\}$, *then* F *has at least one type* (j) *weak* C*-saddle point.*

References

[1] K. Fan, *Minimax Theorems*, Proc. Nat. Acad. Sci., **39** (1953), 42–47.
[2] D. T. Luc, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems, **319**, Springer, Berlin, 1989.
[3] D. Kuroiwa, T. Tanaka, and T. X. D. Ha, *On cone convexity of set-valued maps*, Nonlinear Anal., **30** (1997), 1487–1496.
[4] I. Kuwano, T. Tanaka, and S. Yamada, *Characterization of nonlinear scalarizing functions for set-valued maps.* in Nonlinear Analysis and Optimization, S. Akashi, W. Takahashi, and T. Tanaka (eds.), Yokohama Publishers, Yokohama, 2009, pp.193–204.
[5] I. Kuwano, T. Tanaka, and S. Yamada, *Inherited properties of nonlinear scalarizing functions for set-valued maps*, to appear in the proceedings of the Sixth International Conference on Nonlinear Analysis and Convex Analysis, S. Akashi, Y. Kimura, and T. Tanaka (eds.), Yokohama Publishers, Yokohama, 2010.
[6] I. Kuwano, T. Tanaka, and S. Yamada, *Unified scalarization for sets and set-valued Ky Fan minimax inequalities*, to appear in J. Nonlinear and Convex Anal., 2010.
[7] I. Kuwano, T. Tanaka, and S. Yamada, *Existence Theorems for Cone Saddle Points of Set-Valued Maps*, submitted to J. Optim. Theory Appl.
[8] I. Kuwano, T. Tanaka, and S. Yamada, *Saddle point theorems for set-valued maps via unified scalarization methods*, submitted to Journal of Nonlinear Analysis and Optimization.
[9] M. Sion, *On general minimax theorems.* Pacifc J. Math., **8** (1958), 295–320.
[10] Y. Sonda, I. Kuwano, and T. Tanaka, *Cone-Semicontinuity of Set-Valued Maps by Analogy with Real-Valued Semicontinuity*, to appear in Nihonkai Math. J., 2010.
[11] T. Tanaka, *Generalized Semicontinuity and Existence Theorems for Cone Saddle Points*, Appl. Math. Optim., **36** (1997), 313–322.